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Reexamination of Josephson junctions coupled to transmission lines

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The effect of dissipation when due to the load of a transmission line coupled to a Josephson junction is reconsidered and evaluated by means of a simple direct procedure that supplies analytical expressions. The results are in good agreement with the ones previously reported in the literature. A simple criterion for testing experimental results is introduced.

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Since the 1980s, several works have been devoted to the theme of macroscopic quantum tunneling (MQT). However, a clear interpretation of the results and of their obtainment has not always been given. As is well known, the decisive quantity in determining tunneling behavior is given by the Euclidean action, in which the contribution due to dissipative effects plays an important role. The purpose of the present work is that of once again evaluating this contribution.

With reference to the problem under consideration—an important case of MQT susceptible to experimental verifications and applications—the most satisfying work is, in our opinion, the one by Chakravarty and Schmid [1] in which, by using a sophisticated and elegant method based on the Green's function, results were obtained also in agreement with the so-called "Leggett's prescription" [2]. The application of this prescription makes it possible to obtain directly the same results as the method adopted in [1], thus (apparently) decreasing its importance. This is not completely true, however, since obtaining the above-mentioned result is not completely lacking in points that still need to be clarified. To this end, it is worth recalling that, in a more recent work [3], the debated question of the suppression of tunneling being due to dissipation [4] was reconsidered. The authors showed that the virtual mixing of excited states induced by dissipative interactions tends *to enhance* the tunneling rate in double-well potentials. More in general, the tunneling effect depends greatly on the choice of counterterms introduced in the Lagrangian of the systems considered [2,5], while the approach in [1] is lacking in this exigency.

Typically, most of the works dedicated to this subject are rather complicated [6]. In particular, the work of Ref. [1] is somewhat hermetic as well as concise. Two other works [7,8] are devoted to a better understanding of the procedure and are in agreement with the results of [1]. A drawback in [1] is represented by the choice of an unusual geometry, with the junction situated in the middle of an open transmission line: a more suitable and practical geometry would require the junction to be put at one end of the said line. Another paper is dedicated to the same problem [9], but adopts a different method, based on Laplace's transforms. By using the artifice of halving the bounce time, the results obtained were in substantial agreement with the previous ones. After due consideration, however, this use did not seem at all satisfactory for the reason that halving the bounce time in Ref. [9] was forced by the procedure there adopted, which works in a temporal semispace. Therefore a different evaluation, based on a relatively clearer procedure, is presented here. While the above-mentioned articles [1–3,7,8] are based on functional integrations, the present approach avoids the use of this "delicate" instrument while obtaining the same results [10].

Let us consider an open transmission line coupled to the junction (see Fig. 1). The Lagrangian density can be written in terms of magnetic field storage energy minus electric field storage energy (even if other determinations are admissible) [1,5,11]:

$$\frac{1}{2}\sigma I^2(z,\tau) - \frac{1}{2}\rho V^2(z,\tau), \tag{1}$$

where σ and ρ denote the inductance and capacitance per unit length, respectively; z is the spatial coordinate, and τ is the real time. The current I and the voltage V at the input of the line are functions of τ and are related by the relation [12] $V/I = -iZ_0 \cot(\omega\tau)$, where $Z_0 = (\sigma/\rho)^{1/2}$ is the characteristic impedance of the line and ω is the angular frequency component of the propagating wave. According to the bounce formalism, we now replace ω with $-i|\omega|$ [1], so that $V/I = Z_0 \coth(\omega\tau)$. By substituting this into (1) the Lagrangian density of the line, as seen by the junction at $z=0$, is given by

$$\frac{1}{2}\rho V^2(\tau)[\tanh^2(\omega\tau) - 1]. \tag{2}$$

Then, by changing the variable $\omega\tau$ into $-kz$ according to the propagation condition of a pulse $V(kz-\omega\tau)$ [13] and integrat-

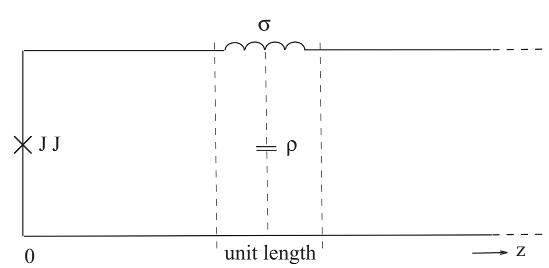

FIG. 1. Josephson junction (JJ) coupled to an open transmission line, where σ and ρ represent the inductance and capacitance per unit length. The maximum of z will be put equal to L.

ing, the complete Lagrangian for a line of length L is given by summing over the ensemble of lines of length $0<z'<L$, thus obtaining

$$\mathcal{L}(\tau,L)=\frac{1}{2}\rho V^2(\tau)\int_0^L dz'[1-\tanh^2(kz')]=\frac{1}{2}\frac{V^2(\tau)}{\omega Z_0}\tanh(kL), \tag{3}$$

where $k=\omega(\sigma\rho)^{1/2}$ is the wave number, with $(\sigma\rho)^{-1/2}$ being the wave velocity. Moreover, according to the second Josephson equation (the line and the junction being coupled at $z=0$), we have $V(\tau)=(\Phi_0/2\pi)\dot{\varphi}(\tau)$, where Φ_0 is the flux quantum and $\dot{\varphi}(\tau)$ is the time derivative of the bounce trajectory $\varphi(\tau)=\varphi_B\,\mathrm{sech}^2(\Omega\tau/2)$, where φ_B is the bounce amplitude and Ω is the plasma frequency of the junction [9,14]. After the substitutions Eq. (3) can be written as

$$\mathcal{L}(\omega,\tau,\tau_0)=\frac{1}{2}\frac{\eta}{\omega}[\dot{\varphi}(\tau)]^2|\tanh(\omega\tau_0)|, \tag{4}$$

where $\eta=(\Phi_0/2\pi)^2/Z_0$ is the "dissipative constant" due to the load of the transmission line and $\tau_0=kL/\omega$ is the delay time of the line. The line, even if considered as ideal without inherent losses, represents a load for the junction, a load that is purely resistive, equal to the characteristic impedance Z_0 of the line when $\omega\tau_0\gg 1$, while it is capacitive, equal to the reactance $(\omega C)^{-1}$, where $C=\rho L$ is the total capacitance, when $\omega\tau_0\ll 1$ [1].

To evaluate the variation of the action integral due to the line, we can follow two different paths. One less accurate result can be obtained by integrating Eq. (4) in time for a fixed value of $\omega=\Omega$. In this way, we have a result that is only partially correct [15]. The other way is based on Fourier transformation and implies integration in frequency, which supplies more accurate results. According to this last procedure, by taking into account that $|V(\omega)|=(\Phi_0/2\pi)\omega\xi(\omega)$, where $\xi(\omega)$ is the Fourier transform of the trajectory [9], the variation of the bounce action, due to the line, is given by

$$\Delta S_B=\frac{\eta}{2}\int_{-\infty}^{\infty} d\omega|\xi(\omega)|^2\omega\tanh(\omega\tau_0). \tag{5}$$

This result, which is exactly equal to the one in Ref. [1] [see Eqs. (16) and (20), there], has been here obtained by a relatively simpler procedure.

By substituting into (5) the explicit form of ξ,

$$\xi(\omega)=\varphi_B\frac{2\sqrt{2}}{\sqrt{\pi}\Omega}\left(\frac{\pi\omega}{\Omega}\right)\mathrm{csch}\left(\frac{\pi\omega}{\Omega}\right), \tag{6}$$

we obtain the following expression for the adimensional function $f=\Delta S_B/\eta\varphi_B^2$ [16]:

$$f(\Omega\tau_0)=\frac{8}{\pi^3}x^4\int_0^{\infty} d\xi\,\xi^3\tanh(\xi)\mathrm{csch}^2(x\xi),$$

$$\xi=\omega\tau_0,\quad x=\frac{\pi}{\Omega\tau_0}. \tag{7}$$

This integral, which represents the solution to our problem, can be evaluated analytically by adopting an expansion of $\tanh(\omega\tau_0)$ in a power series or in a series of exponentials. The power series turns out to be suitable for small values of $\Omega\tau_0=\pi/x$; a careful examination of the integrand in (7) shows that the power-series expansion (converging for $\omega\tau_0<\pi/2$) can be used for $\Omega\tau_0<2$. In this case, the well-known formula [17]

$$\int_0^{\infty} d\xi\,\xi^{2m}\,\mathrm{csch}^2(x\xi)=\frac{\pi^{2m}}{x^{2m+1}}|B_{2m}|, \tag{8}$$

where B_{2m} are the Bernoulli numbers, leads to an asymptotic power series in $(1/x)$. Using the known properties of this kind of series, it is easily found that the behavior of the function (7) is well described (within a few percent), for $\Omega\tau_0<1$, by the expansion up to the fourth term:

$$f(\Omega\tau_0)\approx 0.267(\Omega\tau_0)-0.063(\Omega\tau_0)^3+0.036(\Omega\tau_0)^5 -0.033(\Omega\tau_0)^7. \tag{9}$$

It was more difficult to find an analytical expression in the opposite limit of large values of $\Omega\tau_0$. By using the expansion of $\tanh(\omega\tau_0)$ in a series of exponentials we obtain

$$f(\Omega\tau_0)=\frac{8}{\pi^3}x^4\int_0^{\infty} d\xi\,\xi^3\Big[1+2\sum_{n=1}^{\infty}(-1)^n e^{-2n\xi}\Big]\mathrm{csch}^2(x\xi). \tag{10}$$

Here, the first integral in parentheses can be directly evaluated as [17]

$$\int_0^{\infty} d\xi\,\xi^3\,\mathrm{csch}^2(x\xi)=\frac{1}{x^4}\frac{1}{4}\Gamma(4)\zeta(3). \tag{11}$$

When substituted into (10), this result supplies the value in the limit of $\Omega\tau_0\to\infty$, that is, $f\approx 0.465$, since $\zeta(3)\approx 1.202$. As for the subsequent terms in Eq. (10), the general one is (performing an evident change of variable)

$$\frac{16}{\pi^3}(-1)^n\int_0^{\infty} d\xi\,\xi^3 e^{-2n\xi/x}\,\mathrm{csch}^2(\xi) \tag{12}$$

and an integration by parts gives ($\beta=2n/x$)

$$\int_0^{\infty} d\xi\,\xi^3 e^{-\beta\xi}\,\mathrm{csch}^2\xi=\int_0^{\infty} d\xi(3\xi^2-\beta\xi^3)e^{-\beta\xi}\coth\xi. \tag{13}$$

This integral is known [18] and can be put in the form

$$g(\beta/2)=1.5\left[\zeta\left(3,\frac{\beta}{2}\right)-\frac{\beta}{2}\zeta\left(4,\frac{\beta}{2}\right)\right], \tag{14}$$

where $\zeta(y,\beta/2)$ is the generalized Riemann's zeta function [19]. By substituting this into (10), we eventually obtain

$$f(\Omega\tau_0)=\frac{12}{\pi^3}\zeta(3)+\frac{16}{\pi^3}\sum_{n=1}^{\infty}(-1)^n g\left(\frac{n}{x}\right), \tag{15}$$

which, by using the definition of $\zeta(y,\beta/2)$ [19], can also be written as

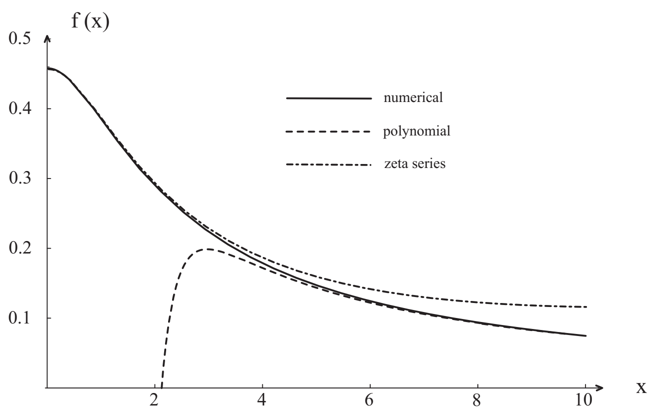

FIG. 2. Plot of the functions described in the text, as a function of $x=\pi/\Omega\tau_0$. The solid line gives the exact value of f in Eq. (7), obtained by numerical integration; the dashed line represents the expression (9), which holds for $x>\pi$, and the dotted one represents the expression (15), stopped to $n=10$, which holds for $x<\sqrt{6}$.

$$f(\Omega\tau_0) = \frac{12}{\pi^3}\zeta(3) + \frac{24}{\pi^3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^n m}{(m+n/x)^4} \qquad (16)$$

to permit a direct calculation. This equation resolves our problem for high values of $\Omega\tau_0$.

In Fig. 2 the plots of functions given by Eq. (15), stopped to $n=10$, and by Eq. (9) are shown, together with the exact value of f, obtained by numerical integration [20]. It must be observed that the series (15) converges for every value of x, provided that a sufficient number of terms is considered in the sum. Using the expansion of $\zeta(y,\beta/2)$ for large values of β [21], it is possible to state that, in order to have an error less than ε, the expansion (15) must include terms up to $n \approx x/\sqrt{6\varepsilon}$. For example, for $\Omega\tau_0=\pi$, stopping to $n=4$ in (15), we obtain $f(\Omega\tau_0=\pi)\approx 0.395$, which is in good agreement (within 2%) with the exact value of Eq. (7).

We have therefore demonstrated that the results obtained are in agreement with those of Refs. [1,2,7,8], although they were obtained here by means of a clearer and simpler procedure. In addition, we note that the geometry adopted, as in Refs. [7,8], is a more suitable one for the experimental testing of these systems. To this end, an approximate relation, based on the shortening of the semiclassical traversal time τ_S as being due to dissipative effects (represented in the actual case by the line load), was given in Ref. [9]. It can be written as $\Delta\tau_S/\tau_S \approx (2/3)\Delta S_B/S_B$, where ΔS_B, for $\Omega\tau_0 \gg 1$, can be assumed to be given by $0.465\eta\varphi_B^2$, while $S_B=(8/15)M\Omega\varphi_B^2$, with M being the "mass" of the particle given by $M = C_J(\Phi_0/2\pi)^2$ and C_J the capacitance of the junction. After substitutions, we find that $\Delta\tau_S/\tau_S$ is now given by [22]

$$\frac{\Delta\tau_S}{\tau_S} \approx \frac{0.58}{Q}, \qquad (17)$$

where $Q=\Omega Z_0 C_J$ is the "quality" factor of the system. According to the results and parameter values of Ref. [23], in which a tunneling (decoupling) time of 78 ps was measured, by comparing it with the half period (π/Ω) in harmonic approximation of 85 ps (assumed as the semiclassical tunneling time in the absence of load), we obtain $\Delta\tau_S/\tau_S=7/85 = 8.2\%$. Considering that in this case, for a sufficiently long τ_0 (say, $\tau_0 \approx 200$ ps, while $\Omega=3.7\times 10^{10}$ s^{-1}), so that $\Omega\tau_0 \approx 7.4$ and $Q=7.2$ ($Z_0=72\ \Omega$, $C_J=2.7$ pF), we obtain $(\Delta\tau_S/\tau_S)Q=0.59$, which is in good agreement with the value predicted by Eq. (17) [24].

In conclusion, we have presented a relatively simple and direct way for evaluating the increment of the bounce action as being due to the load constituted by a transmission line coupled to a Josephson junction. This quantity is of crucial importance for determining tunneling rate. Analytical expressions for the action variation are given in both limits of small and large values of $\Omega\tau_0$. A simple, approximate expression for testing experimental results is also given.

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[10] Also in Ref. [9] the functional integration was not used. Moreover, one is remanded to this work for a discussion and a comparison of different approaches and relative results.
[11] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1950), p. 45.
[12] See, for instance, F. E. Terman, *Electronic and Radio Engineering* (McGraw-Hill, New York, 1955), Chap. 4.
[13] This position certainly holds when $\omega\tau_0 \gg 1$; see Ref. [8]. In any case it will be adopted in order to have (according to the physical evidence) positive values for the action variation.
[14] P. Fabeni, D. Mugnai, G. P. Pazzi, F. Pignatelli, A. Ranfagni, R. Cristiano, L. Frunzio, and L. S. Schulman, J. Supercond. **12**, 829 (1999). There, and here, the assumption is made that dissipation does not change appreciably the bounce trajectory relative to the isolated junction.
[15] The result of this procedure can be expressed as $\Delta S_B/\eta\varphi_B^2 = 0.267 \tanh(\Omega\tau_0)$ which in the limit of $\Omega\tau_0 \ll 1$ is coincident

with the result then obtained by Eq. (7), while in the opposite limit of $\Omega\tau_0 \gg 1$, the value obtained (0.267) is sensibly smaller then the one of Eq. (7); see below.

[16] The following result is similar to the one reported by A. Ranfagni, P. Fabeni, G. P. Pazzi, A. Agresti, G. Viliani and R. Ruggeri, Phys. Rev. B **74**, 195107 (2006), where Eq. (7) gives twice the values of the present Eq. (7), since in that case the tunneling system—an impurity center in alkali-halide crystals—is symmetrically coupled to a double ionic chain.

[17] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1980), p. 352.

[18] See Ref. [17], p. 360.

[19] See Ref. [17], p. 1073.

[20] See, for instance, the dotted curves reported in Fig. 3 of Ref. [9].

[21] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer, Berlin, 1966), p. 25.

[22] For the reason mentioned before, the numerical factor reported in Ref. [9] was approximately twice that of Eq. (17). In spite of this, the agreement with other experimental results quoted in Ref. [9], in addition to ones of Ref. [10], turns out to be again rather acceptable, even if with a greater margin of uncertainty.

[23] D. Esteve, J. M. Martinis, C. Urbina, E. Turlot, and M. H. Devoret, Phys. Scr. **T29**, 121 (1989), where a practical configuration of a junction coupled to a transmission line is shown.

[24] More exactly, for $\Omega\tau_0 \approx 7.4$, $\Delta S_B/\eta\varphi_B^2 \approx 0.446$ and the numerical factor in Eq. (17) should be substituted by 0.56, which can be considered again in reasonable agreement with the value of 0.59 obtained from the experiment.